Strict Finitism Refuted?

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Abstract: In his paper 'Wang's paradox', Michael Dummett provides an argument for why strict finitism in mathematics is internally inconsistent and therefore an untenable position. Dummett's argument proceeds by making two claims: (1) Strict finitism is committed to the claim that there are sets of natural numbers which are closed under the successor operation but nonetheless have an upper bound. (2) Such a commitment is inconsistent, even by finitistic standards.

In this paper I claim that Dummett's argument fails. I question both parts of Dummett's argument, but most importantly I claim that Dummett's argument in favour of the second claim crucially relies on an implicit assumption that Dummett does not acknowledge and that the strict finitist need not accept.

§1. Introduction

According to constructivism in mathematics, 'the meaning of all terms, including logical constants, appearing in mathematical statements must be given in relation to constructions which we are capable of effecting, and of our capacity to recognise such constructions as providing proofs of those statements'[1]. Strict finitism (henceforth SF) is one version of constructivism that takes the phrase 'we are capable' quite literally: we are capable of effecting a construction or surveying a proof if and only if it is in practice within our capacity to do so. So for example, according to SF, a so-called proof of 10^{100} steps cannot truly count as a proof because, although finite, we are in practice unable to survey its details and recognise it as such.

According to Dummett strict finitists are committed to an allegedly absurd view according to which there are non-empty sets of natural numbers that are closed under the successor operation but such that they nonetheless have an upper bound. Otherwise put,

[1] Dummett (1975), p.301.

this view claims that there are sets of natural numbers which are in some sense both 'infinite' (because one can always proceed from a member of the set to its successor and remain within the set) and 'finite' (because if *m* is an upper bound for the set it contains at most *m*-1 members)[2]. To simplify things, let us call set of natural numbers S an IF-set ('infinite finite set') if the following two conditions hold (with the quantifiers ranging over natural numbers)[3]:

(IF-1) $\forall n(n\in \mathrm{S}\rightarrow n+1\in \mathrm{S})$

(IF-2) $\exists n(n\notin \mathrm{S} \wedge \exists m<n(m\in \mathrm{S}))$

Dummett's argument against SF essentially consists of two claims.

(claim-1) SF is committed to the existence of IF-sets.

(claim-2) Such a commitment is inconsistent, even by strict finitistic standards.

I shall take issue with Dummett's argument in favour of claim-1 in **§2** and with his argument in favour claim-2 in **§3**.

§2 Questioning claim-1

Here is Dummett's chief example of an IF-set to which the strict finitist is allegedly committed[4]: Call a natural number *n* apodictic if there exists a finitistically acceptable proof which includes at least *n* steps. According to Dummett the finitist should accept the following two claims:

(1) For any *n*, if *n* is apodictic then *n*+1 is apodictic.

(2) There is a number *M*, such that *M* is not apodictic, but some smaller number than it is apodictic.

If Dummett is correct, then by definition, the strict finitist is committed to the claim that the set of apodictic numbers is an IF-set.

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[2] See Dummett (1975), p.312.

[3] Note that I have replaced the claim that the set has an upper bound with the claim (IF-2). Given that S is non-empty, this claim is weaker than the original and will suffice for the purposes this paper.

[4] See Dummett (1975), p.306.

So, is Dummett correct? Dummett's claim (2) seems perfectly correct as long as the finitist allows for some liberty in the methods we can use for the construction of numbers. If the construction of numbers via exponentiation is allowed, then M=10^{100} is a constructible (and hence a finitistically acceptable) number, is clearly not apodictic, and is larger than some apodictic number (for example, the number three). Claim (1), however, is more dubious: why should the finitist accept it? An initial thought might be this: surely, if we can construct a proof of *n* steps, then we can add to our proof just one more step.

Anyone who has had previous dealings with the sorites paradox should know that the temptation to accept this initially compelling argument should be resisted. This is particularly true for the kind of predicate in question. Consider the following two claims:

- **(A)** If *n* grains do not make a heap, then *n*+1 grains do not make a heap.
- **(B)** If placing *n* grains on the scale will not tip the balance, the placing *n*+1 grains on the scale will not tip the balance.

Given any value of *n*, both claims seem initially compelling. But there is a difference: While there might well be (as the epistemicist about vagueness claims) a value of *n* for which claim (A) is false, this is a matter of subtle philosophical dispute. But that there is a value of *n* for which (B) is false is an undisputed physical fact: if we keep piling grains of sands on one scale of a balance, the total mass of the grains will eventually add up to a larger mass than is present on the other scale, hence tipping the balance. No non-standard semantic theory will help maintain the intuitive plausibility of (B).

Now it seems to me that Dummett's case of 'apodictic' resembles case (B) more than case (A). It is true that there is some vagueness in the notion of being a 'surveyable in practice' proof, and hence in the notion of being a finitistically acceptable proof. But it also seems true that however we precisify this notion, there will be a sharp yet difficult to detect limit to our ability to survey a proof. Subtle physical facts such as the speed of our reading and the length of our lives will limit the length of the proofs which are surveyable. It seems to me that the initial appeal of the tolerance claim (1) stems less from the vagueness of 'apodictic' and more from a consideration such as in claim (B): How can adding just one teeny-weeny grain tip the balance? How can adding just one other teeny-weeny step to the proof make it beyond our reach? If I am correct about this then regardless of the view she takes on vagueness, the strict finitist will reject (1), and hence will not be committed to the claim that the set of apodictic numbers is an IF-set.[5]

Nevertheless, in the following section I shall grant Dummett the assumption that the strict finitist is committed to the claim that the set of apodictic numbers is an IF-set, and I shall argue that Dummett fails to prove that such a commitment is inconsistent.

§3 Questioning claim-2

Putting aside apodictic numbers for a moment, one might think that a commitment to the existence of *any* IF-set is obviously inconsistent. Here's why: let S be an IF-set. By IF-2, S is non-empty. Let m_0 be the least number in S. By IF-1, $\forall k(k\in S\rightarrow k+1\in S)$. So by induction (using m_0 as the base case) for any natural number $n\geq m_0$, $n\in S$. But this contradicts IF-2, so we reach a contradiction.

The trouble with this argument is that there are reasons to believe that a strict finitist should not accept the principle of induction. This is so because according to SF not every number can be 'reached' by starting from zero and repeatedly applying the successor operation. The number series is in this sense 'gappy': ${}_{10}10^{100}$ is a finitistically acceptable number but there are (according to a platonist) smaller numbers than it which the strict finitist does not acknowledge. So the general appeal to induction in order to show that the commitment to any IF-set is inconsistent is not a good strategy.

Dummett's argument in favour of claim-2 avoids this problem by choosing a different strategy. Instead of trying to show that the commitment to *any* IF-set is inconsistent, Dummett argues that there are *particular* sets, which the strict finitist is committed to

[5] A somewhat similar point is made by Wright (1993), p.161.

claiming are IF-sets, and such that this commitment entails (by a finitistically acceptable notion of entailment) a contradiction.

We can start with the set of apodictic numbers which I have taken for the sake of argument to be (according to SF) an IF-set. Instead of appealing to induction, we can consider a step-by-step argument, such as the following:

(1) One is apodictic.

(2) If one is apodictic then two is apodictic (by IF-1, and Universal Instantiation). So two is apodictic (by (1) and Modus Ponens).

(3) If two is apodictic then three is apodictic. So three is apodictic (by (2) and MP).

…

(2^{100}) If 2^{100}-1 is apodictic then 2^{100} is apodictic. So 2^{100} is apodictic.

Clearly, the strict finitist holds that 2^{100} is not apodictic, so accepting the last step of this argument leads to a contradiction. Since plausibly, the strict finitist accepts UI and MP she should accept *each step* in the above argument. However, the argument *as a whole* is not finitistically acceptable because it contains too many steps. Had we attempted to fully spell out the argument it would be too long to be surveyable.

Anticipating this problem, Dummett suggests a more sophisticated example. Instead of considering the set of apodictic numbers, consider the set of 'small' numbers. A number *n* is said to be small if *n*+100 is apodictic. It is easy to see that, granting that the strict finitist takes the set of apodictic numbers to be an IF-set, she will also take the set of small numbers to be an IF-set: If *n*+100 is apodictic, then *n*+100+1 is apodictic, so IF-1 holds. And since 101 is apodictic, 1 is small. But 10^{100}>1 and is not small, so IF-2 holds.

Dummett continues to argue as follows: 'Now it seems reasonable to suppose that we can find an upper bound *M* for the totality of apodictic numbers such that *M*-100 is apodictic. (If this does not seem reasonable to you, substitute some larger number *k* for 100 such that it does seem reasonable…and understand *k* whenever I speak of 100)'[6].

[6] Dummett (1975), p.306.

We can now construct a parallel argument to the one suggested above:

(1) One is small.

(2) If one is small then two is small (by IF-1 and UI). So two is small (by (1) and MP).

(3) If two is small then three is small. So three is small (by (2) and MP).

…

(*M*-100) If *M*-100-1 is small then *M*-100 is small. So *M*-100 is small.

The last step of this argument contradicts the stipulation that *M* is not apodictic, because if *M*-100 is small then *M*-100+100=*M* is apodictic. By the same reasoning as above, the strict finitist should accept each step of this argument. But more interestingly, this time the strict finitist should accept the proof *as a whole*: the proof is only *M*-100 steps long[7], and by stipulation *M*-100 is apodictic so the proof should be short enough to be acceptable[8]. So it seems that the strict finitist is finally driven to a contradiction.

Or is she? I would now like to claim that Dummett's argument contains a crucial hole. The crux of the problem lies in the supposedly naïve side-comment in parentheses which I quoted above: '(If this does not seem reasonable to you, substitute some larger number *k* for 100 such that it does seem reasonable…and understand *k* whenever I speak of 100)'. I agree that there must be *some* number *k* that satisfies Dummett's constraints (namely: there is a number *M*, such that *M* is not apodictic, but *M*-*k* is apodictic). But suppose we choose for *k* a number that is itself not apodictic. Recall that '*n* is small' will now mean '*n*+*k* is apodictic'. So if *k* is not apodictic, then '*n* is small' would be false even for *n*=1. So the first premise of the above argument ('one is small') will be false, and the argument

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[7] If you are worried that each of lines I note actually counts as two steps in the proof, replace the definition of '*n* is apodictic' with 'there exists a finitistically acceptable proof which includes at least 2*n* steps'.

[8] Actually, one should note that on Dummett's definition of 'apodictic' this does not quite follow. According to Dummett, *n* is apodictic if there is *some* finitistically *n*-step long proof. This still leaves open the possibility that there are other *n*-step long proofs are not finitistically acceptable (for example because each of the steps in such proofs are quite long). This problem can be solved by amending the definition of '*n* is apodictic' to say something like 'Any proof which contains *n* steps which are individually finitistically acceptable is finitistically acceptable as a whole'.

will not go through. For Dummett's argument to work, the finitist must accept the following:

(*) There is a number k and a number M, such that M is not apodictic, k is apodictic, and M-k is apodictic.

But why should the finitist accept (*)? Try to think of examples. 10^{100} is not apodictic, and it is easy to come up with examples for a number k such that $10^{100}-k$ is apodictic. But any example that immediately springs to mind is something like $k= 10^{100}-100$, which is not an apodictic number. It is thus at least not obvious that we can come up with an example that would vindicate (*).

Here is one suggestion on how one could defend (*). Take an apodictic number k. If $k+k=2k$ is not apodictic, then (*) is proved (letting k and $2k$ stand for k and M in (*)). Otherwise, take $2k+2k=2^2k$. If it is not apodictic, then (*) is proved (with respect to $2k$, $4k$). Otherwise, take $4k+4k$… And so forth. Now at some point, you are bound to get to a number that is not apodictic. Moreover, since 2^{100} is not apodictic, you will get to this number in 100 or fewer steps, so the strict finitist cannot claim that the above argument will be spelled out as a proof that is too long.

As strong as this amended argument might seem, I don't think it works. For I take it that like most constructivists the strict finitist will adopt something like an intuitionistic interpretation of the quantifiers and connectives. In particular, she would interpret the existential quantifier in (*) so that one should accept (*) only if one can come up with *particular* values of k and M that satisfy the claim. A general disjunctive argument for their existence (as outlined above) is insufficient.

But cannot one simply follow the 100 steps suggested above until one comes up with a particular value of k as necessary? I think the answer to this is that one cannot. The reason is that although (by platonistic standards) there exists a number k such that k is apodictic but $2k$ is not, we may not *know* for which number k this happens. To put it otherwise: if 'apodictic' is, as Dummett claims, truly vague then the predicate 'apodictic' is undecidable in the sense that there are some numbers n for which we do not know and are not in a position to know either that they are apodictic or that they are not apodictic[9]. The finitist would agree 2^0k is apodictic, and that $2^{100}k$ is not apodictic, but why should she accept that every member of $\{2^nk: 0 \leq n \leq 100\}$ is either apodictic or not apodictic?

The suggested argument in favour of (*) thus resembles the following proof for the claim that there are two irrational numbers a, b such that a^b is rational: if $\sqrt{2}^{\sqrt{2}}$ is rational, take a=b=$\sqrt{2}$, and we are done. Otherwise take a=$\sqrt{2}^{\sqrt{2}}$ and b=$\sqrt{2}$, which yields $\sqrt{2}^{\sqrt{2}*\sqrt{2}}$=2. Q.E.D. This proof is a famous example of the kind of proof that is unacceptable by intuitionistic standards.[10] This is because the proof presupposes that every number is either rational or irrational even though 'rational' is an undecidable predicate. The above argument in favour of (*) presupposes that, in spite of the fact that 'apodictic' is undecidable, every number is either apodictic or not apodictic. But this is an assumption that the strict finitist will not grant.

What the above argument *does* show is that the strict finitist cannot claim that whenever k is apodictic, and l is apodictic then k+l must be apodictic. So if I know k and l to be apodictic, I am not thereby in a position to know that k+l is apodictic. But it does not follow that there is any case where I know that k and l are both apodictic, and that k+l is not apodictic (I might simply be agnostic as to whether k+l is apodictic). And if there are no such cases of k and l, then the strict finitist is not forced to accept (*). Even if we (as platonists) assume that 'apodictic' has a sharp cut off point, we might concede that that the gap between the 'knowably apodictic' numbers and the 'knowably not apodictic' numbers is so large that one cannot bridge across it using an apodictic number. The strict

[9] Note that this view will be shared by most theories of vagueness: let '*F*' be a vague predicate, and led a be borderline case of F. Since knowledge is factive, views (such as supervaluationism or fuzzy logic) which take both '*Fa*' and '¬*Fa*' to be not true must hold that neither claim is knowable. And views such as epistemicism which holds that one of the two claims is true, also typically holds that we are not in a position to know which of the two claims it true.

[10] See Dummett (1977), p.10.

finitist is thus not forced to accept (*), and Dummett's argument fails to establish claim-2.[11]

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